Annual Risk Measures and Related Statistics

A common method to annualize monthly risk measures is to multiply the outcome by 12 or the square root of 12. Paul D. Kaplan has shown that for one particular measure, the standard deviation, this approach is incorrect, given that returns are compounded over time rather than summed. Kaplan has also presented a method to calculate the annual standard deviation correctly. This article expands on Kaplan's work. It shows for a wide palette of risk measures and related statistics how the annual variants can be calculated correctly, under the assumption of compounded returns.

Arno E. Weber, CIPM
is Agile software test analyst at Ortec Finance in the Netherlands. In this role, he is responsible for the development of the PEARL software solution for investment performance measurement and attribution. Prior to joining the development team, Arno was a business consultant responsible for the implementation of PEARL. He holds a Master's degree in Mathematics from the Free University of Amsterdam, where he graduated cum laude in 2007.

INTRODUCTION

A commonly used method to annualize risk measures based on monthly returns is to multiply the outcome by 12 or $\sqrt{12}$, depending on the type of measure. This way, the measure should be expressed in the same unit as the annual return. Paul D. Kaplan has shown[1] that for standard deviation, this simple approach is mathematically *incorrect*, given that returns are aggregated over time by compounding rather than summing. Kaplan has also presented a method which calculates the annual standard deviation correctly, under the assumption that returns are compounded over time.

I agree with Kaplan's arguments and his proposed method, however, his analysis is limited to one measure, the standard deviation. For other risk measures, the analysis is either not straightforward or it results in different formulas.

This article expands on Kaplan's work. The main purpose is to show for a wide palette of risk measures and related statistics, how the annual variants can be calculated correctly, under the assumption of compounded returns.

The article is set up as follows. First, I will give a mathematical background on probability distributions of returns and introduce three "basic measures" related to a return distribution: the monthly *mean*, the monthly *standard deviation*, and the monthly *covariance* of a portfolio and its benchmark. For these three monthly basic measures, I will show how the measure is commonly estimated based on realized monthly returns. The reason to focus on these measures first is that many other risk measures and related statistics can be derived from these three basic measures.

In the next section, I'll recall how the annual variants of the three basic measures are estimated using the common but incorrect method (assuming summed returns). Then, I will show how the annual variants of these measures can be correctly estimated (assuming compounded returns). Two methods are presented: one which results in closed formulas, and one which involves Monte-Carlo simulation: bootstrapping.

Given these two estimation methods for the annual basic measures, I will show that various risk measures and related statistics are functions of the basic measures. This implies that estimating annual variants of these statistics becomes straightforward once estimations of the basic measures are available.

In the last part of the article, I will give two examples of risk measures for which a closed formula for the annual variant cannot be derived. For these measures, the bootstrap method becomes particularly useful to estimate the annual variants.

Throughout the article, I will illustrate the various calculation methods with numerical examples, which give an impression in what scenarios the available methods give significantly different results.

THE "BASIC MEASURES" AND THEIR MONTHLY ESTIMATES

In this section, I will first give a mathematical background on probability distributions of returns. Next, I will introduce three "basic measures" related to the probability distribution of monthly returns and show how these measures are usually estimated based on a set of realized values from the monthly return distribution.

Suppose that $(R_m, \bar{R}_m)$ is a pair of random variables representing a monthly return of a portfolio, and the return of the benchmark assigned to this portfolio over the same month. When performing risk analysis, a researcher is commonly interested in properties of the probability distribution of $(R_m, \bar{R}_m)$. This probability distribution represents the universe of all possible outcomes of $(R_m, \bar{R}_m)$ (and their probabilities), as opposed to realized outcomes of $(R_m, \bar{R}_m)$.

Three "basic measures" related to the probability distribution of $(R_m, \bar{R}_m)$, which a researcher might be interested in, are the portfolio *mean* μ_m (also called *expectation*), the portfolio *standard deviation* σ_m , and the *covariance* c_m of the portfolio and the benchmark. These measures are mathematically defined as follows:

$$\mu_m := E[R_m] \quad , \quad (1)$$

$$\sigma_m := Std[R_m] = \sqrt{E[(R_m - \mu_m)^2]} \quad , \quad (2)$$

$$c_m := Cov[R_m, \bar{R}_m] = E[(R_m - \mu_m)(\bar{R}_m - \bar{\mu}_m)] \quad . \quad (3)$$

Where, *E*[.] is the expectation of a random variable, *Std* [.]the standard deviation of a random variable, and *Cov*[.,.] the covariance of two random variables. Throughout this article, I will use a notation with upper bar for measures representing the *benchmark*. For instance, the variable $\bar{\mu}_m$ in (3) is the mean of the benchmark and is defined similarly to the portfolio's mean:

$$\bar{\mu}_m := E[\bar{R}_m]$$

The above measures μ_m, σ_m and c_m are properties of the unknown probability distribution of the pair $(R_m, \bar{R}_m)$. In practice, these measures are often estimated based on realized and observed values of the portfolio and benchmark return.

Suppose that a dataset of n pairs of observed monthly portfolio and benchmark returns

$$(r_{m,1}, \bar{r}_{m,1}), \ldots, (r_{m,n}, \bar{r}_{m,n})$$

is available. By assumption, these return pairs are independent realizations from the probability distribution of $(R_m, \bar{R}_m)$. In an ex-post risk analysis, the three basic measures defined above are commonly estimated by calculating the *sample mean* $\hat{\mu}_m$, the *sample standard deviation* $\hat{\sigma}_m$, and the *sample covariance* $\hat{c}_m$, as follows[2]:

$$\hat{\mu}_m := \frac{1}{n}\sum_{i=1}^{n} r_{m,i} \quad , \quad (4)$$

$$\hat{\sigma}_m := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_{m,i} - \hat{\mu}_m)^2} \quad , \quad (5)$$

$$\hat{c}_m := \frac{1}{n-1}\sum_{i=1}^{n}(r_{m,i} - \hat{\mu}_m)(\bar{r}_{m,i} - \hat{\bar{\mu}}_m) \ . \quad (6)$$

Throughout this article, I will denote variables with a circumflex (^) to indicate that the variable represents an *estimate* of the corresponding variable without circumflex. This is a common notation in statistics. Thus, $\hat{\mu}_m$ is an estimate for μ_m , and so forth.

The above estimates $\hat{\mu}_m$,$\hat{\sigma}_m$ and $\hat{c}_m$ of the monthly basic measures are often annualized in order to fairly compare these measures between different portfolios over time periods of possibly different length. I will now explain why the most common annualization method is mathematically incorrect.

THE SIMPLE BUT INCORRECT ANNUALIZATION METHOD

Now suppose that a set of $T := 12$ monthly portfolio returns

$$R_{m,1}, \ldots, R_{m,T}$$

and benchmark returns

$$\bar{R}_{m,1}, \ldots, \bar{R}_{m,T}$$

make up one full year.[3] Also suppose that the return pairs

$$(R_{m,1}, \bar{R}_{m,1}), \ldots, (R_{m,T}, \bar{R}_{m,T})$$

are independent and identically distributed (i.i.d.) pairs of random variables with the same probability distribution as $(R_m, \bar{R}_m)$ (This is a common assumption within the investment management industry.)

If an annual portfolio return R_A equals the *sum* of the T monthly returns:

$$R_A := \sum_{t=1}^{T} R_{m,t} \quad , \quad (7)$$

and similarly for the benchmark, then, according to the laws of probability, the annual mean, standard deviation and covariance are equal to the corresponding monthly measure multiplied by either $\sqrt{T}$ or T as follows:

$$\mu_A := E[R_A] = T \cdot \mu_m \quad , \quad (8)$$

$$\sigma_A := Std[R_A] = \sqrt{T} \cdot \sigma_m \quad , \quad (9)$$

$$c_A := Cov[R_A, \bar{R}_A] = T \cdot c_m \quad . \quad (10)$$

For the interested reader, I provide mathematical proofs of these known equations in Appendix A.

Equations (8)-(10) imply that the three annual basic measures can be simply estimated by multiplying each corresponding monthly estimate by T or $\sqrt{T}$:

$$\hat{\mu}_A := T \cdot \hat{\mu}_m \quad , \quad (11)$$

$$\hat{\sigma}_A := \sqrt{T} \cdot \hat{\sigma}_m \quad , \quad (12)$$

$$\hat{c}_A := T \cdot \hat{c}_m \quad . \quad (13)$$

This approach is widely used in the field of performance measurement, and I will refer to this annualization method as the "simple method."

Is the simple method also correct? *If* returns are summed over time as in (7), *then* the simple method given by formulas (11)-(13) is the correct way to annualize the measures. However, the type of returns that performance measurement usually deals with do *not* sum over time. In fact, the annual portfolio return is made up by *compounding* the monthly portfolio returns:

$$R_A := \prod_{t=1}^{T} (1 + R_{m,t}) - 1 \quad , \quad (14)$$

and similarly for the benchmark.

Because this expression is not a sum, the assumption (7) on which the simple method relies is not valid. Therefore, annualizing risk measures by multiplying by T or $\sqrt{T}$ is mathematically incorrect.

THE CORRECT METHOD

Given that monthly returns compound over time as in (14), it is possible to derive the correct method to calculate annual variants of the basic measures. This results in the following formulas for the correct annual mean μ_A, standard deviation σ_A and covariance c_A :

$$\mu_A = (1 + \mu_m)^T - 1 \quad , \quad (15)$$

$$\sigma_A = \sqrt{(\sigma_m^2 + (1 + \mu_m)^2)^T - (1 + \mu_m)^{2T}} \, , \quad (16)$$

$$c_A = \left(c_m + (1 + \mu_m)(1 + \bar{\mu}_m)\right)^T - (1 + \mu_m)^T (1 + \bar{\mu}_m)^T \quad . \quad (17)$$

See Appendix B for the mathematical proofs of these equations.

It follows that estimates for the annual variants of the three basic measures can be calculated by using the same formulas, after replacing the unknown monthly measures by their estimates:

$$\hat{\mu}_A := (1 + \hat{\mu}_m)^T - 1 \quad , \quad (18)$$

$$\hat{\sigma}_A := \sqrt{(\hat{\sigma}_m^2 + (1 + \hat{\mu}_m)^2)^T - (1 + \hat{\mu}_m)^{2T}} \, , \quad (19)$$

$$\hat{c}_A := \left(\hat{c}_m + (1 + \hat{\mu}_m)(1 + \hat{\bar{\mu}}_m)\right)^T - (1 + \hat{\mu}_m)^T (1 + \hat{\bar{\mu}}_m)^T \quad . \quad (20)$$

At this point, I'd like to pay attention to some properties of these formulas. Note that annualization of *returns* can be viewed as a transformation of one return into another. A monthly realized return r_m is transformed to an annual return r_A by the following formula[4]:

$$r_A := (1 + r_m)^T - 1 \quad . \quad (21)$$

No information other than the monthly return r_m and the number of months in a year T are necessary to calculate the annual return.

For risk measures, the simple annualization method by multiplying by T or $\sqrt{T}$ can also be viewed as a transformation, as it only needs the monthly measure and T as input. This holds for all three basic measures.

Now if you look at formula (18), the correct estimation method for the annual mean, you can see that the method is the same as for annualizing a single return (formula (21)), and, therefore, also a transformation.

For the annual standard deviation (19), on the other hand, you need to know not only the monthly standard deviation and T, but also the monthly mean. It is therefore not a direct transformation of a monthly standard deviation to an annual standard deviation.[5] The fact that the monthly mean has influence on the annual standard deviation was also noted by Kaplan. It has various consequences for practitioners in the investment management industry.

For example, if two portfolios have the same monthly standard deviation, it is possible that their annual standard deviations are different (because their means are different). This can be explained intuitively by the geometric nature of the annual return (see formula (14)). A large mean will blow up the volatility of the annual return, since it is compounded T times. Moreover, a large positive mean will give a relatively large annual standard deviation, while a large negative mean will give a smaller annual standard deviation.

In a manager selection process where portfolio managers are selected by taking into account the risk profile (standard deviation) of their portfolios, it is important to consider whether to compare the monthly or annual standard deviation, or both. In particular, if the monthly standard deviations are the same, a portfolio with negative mean return has a smaller annual risk than a portfolio with positive mean return. In the next subsection, I will illustrate these cases with numerical examples.

For the annual covariance (20), a similar remark as for the standard deviation holds: to calculate the annual variant you need the means of the portfolio and benchmark as additional information next to the monthly covariance and T. Thus, the method is not a transformation. If two portfolio and benchmark pairs have the same monthly covariance but different means, their annual covariance becomes different. This will be illustrated numerically as well in the next subsection.

Numerical Examples

Table 1 defines eight scenarios to illustrate the different calculation methods. These scenarios (a)-(h) are characterized by the values of the monthly estimated basic measures for the portfolio and its benchmark. From those, the scenarios (a) and (b) are most realistic: the means are small, but for (b) the standard deviation is larger. In scenarios (c) and (d) the standard deviations are the same as in (b), but the means have quite large values (positively for (c) and negatively for (d)). Also note that in each scenario (a)-(d) the monthly standard deviation is the same for the portfolio (PF) and benchmark (BM), only the mean differs. The scenario set (e)-(h) contains the same scenarios as (a)-(d), except that the covariance numbers are chosen to result in a high correlation between the portfolio and the benchmark for (e)-(h) compared to (a)-(d).

For these eight scenarios, I have calculated and compared the annual estimations of the basic measures, using both the simple method and the correct method.

Table 1: Scenarios and Their Monthly Estimated Basic Measures

	Scenarios							
Monthly measure	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
PF Mean	0.3%	0.3%	2%	-2%	0.3%	0.3%	2%	-2%
BM Mean	0.2%	0.2%	1.5%	-1.5%	0.2%	0.2%	1.5%	-1.5%
PF/BM Std.deviation	2%	6%	6%	6%	2%	6%	6%	6%
Covariance	0.002%	0.018%	0.018%	0.018%	0.039%	0.356%	0.356%	0.356%

Table 2: Annual Basic Estimated Measures in Various Scenarios

	Scenarios							
Annual PF Mean	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Simple method ($\cdot T$)	3.60%	3.60%	24.00%	-24.00%	3.60%	3.60%	24.00%	-24.00%
Correct method (18)	3.66%	3.66%	26.82%	-21.53%	3.66%	3.66%	26.82%	-21.53%
Annual BM Mean								
Simple method ($\cdot T$)	2.40%	2.40%	18.00%	-18.00%	2.40%	2.40%	18.00%	-18.00%
Correct method (18)	2.43%	2.43%	19.56%	-16.59%	2.43%	2.43%	19.56%	-16.59%
Annual PF Std.deviation								
Simple method ($\cdot\sqrt{T}$)	6.93%	20.78%	20.78%	20.78%	6.93%	20.78%	20.78%	20.78%
Correct method (19)	7.17%	21.69%	26.09%	16.82%	7.17%	21.69%	26.09%	16.82%
Annual BM Std.deviation								
Simple method ($\cdot\sqrt{T}$)	6.93%	20.78%	20.78%	20.78%	6.93%	20.78%	20.78%	20.78%
Correct method (19)	7.09%	21.46%	24.72%	17.78%	7.09%	21.46%	24.72%	17.78%
Annual Covariance								
Simple method ($\cdot T$)	0.02%	0.22%	0.22%	0.22%	0.47%	4.27%	4.27%	4.27%
Correct method (20)	0.03%	0.23%	0.32%	0.15%	0.50%	4.60%	6.38%	2.96%

This comparison is summarized in Table 2. Note that for scenarios (e)-(h) the results are the same as for (a)-(d), except for the annual covariance.

In general, when the mean and standard deviation are small, the results of the simple and correct method are quite close. This is clearly visible in the results of scenarios (a) and (e). For larger values of the mean and standard deviation, the differences between both methods also gets larger.

Scenarios (b)-(d) confirm the behavior of the annual standard deviation as I have explained in the previous subsection. The monthly standard deviations are the same (6%), but the means are different. Now the correct annual standard deviations also become quite different: the value varies between 16.82% and 26.09 percent. Moreover, a large positive mean, like in (c), gives a relatively large annual standard deviation, while a large negative mean, like in (d), gives a smaller annual standard deviation. These effects are less apparent for a small mean.

The above observations confirm that the monthly mean is an important factor for the annual standard deviation, as mentioned before. Any annualization method depending on the monthly standard deviation alone cannot be correct. In the three scenarios (b)-(d) the simple method incorrectly gives the same number 20.78%, since the monthly standard deviation is the same for those scenarios and the simple method only depends on this number.

The above conclusions for the standard deviation can be stated in a similar way for the covariance. At the bottom of Table 2, the annual covariance is compared for the scenarios (a)-(h). The scenarios show that the sizes of the portfolio and benchmark means have significant impact on the value of the annual covariance. Scenarios (c) and (d) (low covariance) and (g) and (h) (high covariance) all show a relatively large difference between the simple method and the correct method. Also, if portfolios have the same monthly covariance, with respect to their benchmarks (like in (b)-(d) and (e)-(h)), then the annual covariance might become significantly different,

Table 3: Dataset of 60 Monthly Equity Portfolio Returns

-8.11%	-7.44%	-4.59%	-4.02%	-3.87%	-3.49%	-2.90%	-2.58%	-2.56%	-2.18%	-2.09%	-1.67%
-1.65%	-1.47%	-1.01%	-0.77%	-0.50%	-0.06%	-0.01%	0.12%	0.23%	0.23%	0.38%	0.45%
0.62%	0.67%	0.70%	0.79%	0.83%	1.26%	1.27%	1.30%	1.40%	1.44%	1.44%	1.50%
1.99%	2.10%	2.24%	2.26%	2.34%	2.35%	2.40%	2.55%	2.57%	2.59%	2.97%	3.26%
3.39%	3.52%	3.61%	3.81%	3.89%	3.94%	4.06%	4.28%	5.22%	6.06%	6.32%	8.96%

depending on the means of each portfolio and its benchmark.

BOOTSTRAPPING

In the above analysis, the annual basic risk measures were estimated using explicit, closed formulas. In this section I will discuss a second method to estimate an annual risk measure based on a series of observed monthly returns. It is a Monte-Carlo simulation method, and is called *bootstrapping*.

Given a dataset of monthly portfolio returns,[6] the bootstrap algorithm works as follows.

1. Randomly select T monthly returns from the dataset of observed monthly returns. Repeat this a large number of times, say, 10,000 times.

2. For each of the 10,000 sets of T returns, combine the T returns to an annual return, either by summing or compounding, depending on your assumptions regarding the way returns combine over time. This gives a sample of 10,000 annual returns.

3. Calculate an estimate for the target measure based on the 10,000 annual returns. For instance, to estimate the annual standard deviation σ_A, calculate the sample standard deviation of the 10,000 annual returns.

Bootstrapping is very useful when only a few, *i.e.*, 5 years of return data is available. Estimating an annual measure directly based on five annual returns is not quite accurate. With bootstrapping you can simulate a large number of annual returns by using the additional information given by the $5 \cdot 12 = 60$ monthly returns. This leads to a more accurate estimation of the annual measure.

<u>Numerical Examples</u>

In the following, I will illustrate the application of the bootstrap method for the annual mean and standard deviation. For these measures, I will show that the bootstrap estimations are very close to the results of the closed formulas (18) and (19). This is expected since the bootstrap method approximates these results.

Consider the dataset of 60 monthly returns in Table 3. This dataset is based on a 5-year monthly return series of a real portfolio invested in equity securities.[7]

Table 4 defines the scenarios (a)-(d) as used before, but this time for just the portfolio (so no benchmark data). Each scenario is defined by its estimated mean and standard deviation. Note that the returns in Table 3 have a (sample) mean and standard deviation 0.84% and 3.12%, respectively. This does not match with any of the investigated scenarios (a)-(d). Therefore, for each scenario (a)-(d) in Table 4, I have created a custom set of 60 monthly returns by taking the returns from Table 3, and scaling all returns to match the sample mean and sample standard deviation given by the scenario.

Table 5 shows estimates of the annual mean and standard deviation after applying four different methods:

- the simple method (multiplying by T or $\sqrt{T}$);
- the bootstrap using summed returns in step 2;
- the correct method (see formulas (18) and (19));
- the bootstrap using compounded returns in step 2.

The two bootstraps in this analysis use a custom set of returns for each of the given scenarios, as described previously. In each bootstrap, 10,000 annual returns are gen-

Table 4: Scenarios and Their Monthly Estimated Measures

	Scenarios			
Monthly measure	(a)	(b)	(c)	(d)
Mean	0.3%	0.3%	2%	-2%
Standard deviation	2%	6%	6%	6%

Table 5: Annual Estimated Measures, Using Various Techniques, in Various Scenarios

	Scenarios			
Annual Mean	(a)	(b)	(c)	(d)
Simple method ($\cdot T$)	3.60%	3.60%	24.00%	-24.00%
Bootstrap (∑)	3.59%	3.57%	23.97%	-24.03%
Correct method (18)	3.66%	3.66%	26.82%	-21.53%
Bootstrap (∏)	3.65%	3.65%	26.81%	-21.53%
Annual Standard deviation				
Simple method ($\cdot\sqrt{T}$)	6.93%	20.78%	20.78%	20.78%
Bootstrap (∑)	6.94%	20.81%	20.81%	20.81%
Correct method (19)	7.17%	21.69%	26.09%	16.82%
Bootstrap (∏)	7.18%	21.75%	26.15%	16.86%

(∑) assuming summed returns
(∏) assuming compounded returns

erated either by summing or compounding the randomly selected monthly returns. Finally, the target measure (mean or standard deviation) is estimated based on the 10,000 annual returns. For this last step the same estimation method as for the monthly estimations in Table 4 is used, for both the mean and the standard deviation.

The resulting numbers show that the simple method and the bootstrap with summed returns are quite close. This is expected given that the simple method assumes that returns are summed over time. Similarly, the correct method and the bootstrap with compounded returns are also quite close. This is also expected because the correct formulas have been derived assuming compounded returns.

The bootstrap is a flexible method, which gives an estimation of the annual variant of any risk measure, also when a closed formula for the annual measure is not available. In a later section, I will show two examples of such measures and how the bootstrap method can be applied on these measures.

For the annual mean and standard deviation, the results of the above bootstraps assuming compounded returns give an intuitive confirmation that the closed formulas (18) and (19) are in fact correct.

DERIVED RISK MEASURES AND RELATED STATISTICS

At this point, this article has presented two estimation methods for the annual *basic* risk measures (mean, standard deviation and covariance); one involving closed formulas and another involving simulation: bootstrapping.

Many other risk measures and risk-related statistics are functions of two or more basic measures, and can, therefore, be derived from the basic measures. For such statistics, it is straightforward to calculate an annual variant using the already calculated estimations of the annual basic risk measures. I will give a few examples below.

Active Risk

A widely used risk measure is *active risk* (also called *tracking error*). Let index p refer to any periodicity of a measure, in particular $p = m$ refers to a monthly measure and $p = A$ refers to an annual measure. For a general periodicity p, the active risk σ_p^{Exc} is defined as the standard deviation of the excess return of the portfolio versus the benchmark:

$$\sigma_p^{Exc} := Std\left[R_p - \bar{R}_p\right] \quad . \quad (22)$$

How can the annual active risk be estimated using monthly portfolio and benchmark returns? At first sight, because the active risk is the *standard deviation* of the excess return, one might think that formula (19) applies using the monthly sample standard deviation and monthly sample mean of the observed *excess* returns (instead of portfolio returns). However, this would not be correct, since formula (19) assumes that the underlying monthly returns compound to an annual return. This is *not* the case for the *excess* returns. Both the portfolio returns and benchmark returns compound to an annual return and *afterwards* the subtraction takes place. Thus, (19) cannot be used directly.[8] Fortunately, one of the laws of probability tells us that active risk (in any periodicity p) can be expressed in the portfolio and benchmark standard deviation, and the covariance, as follows:

$$\sigma_p^{Exc} = \sqrt{\sigma_p^2 + \bar{\sigma}_p^2 - 2 \cdot c_p} \quad , \quad (23)$$

Therefore, the active risk can be estimated by:

$$\hat{\sigma}_p^{Exc} := \sqrt{\hat{\sigma}_p^2 + \hat{\bar{\sigma}}_p^2 - 2 \cdot \hat{c}_p} \quad . \quad (24)$$

An estimate of the annual active risk $\hat{\sigma}_p^{Exc}$ can be calculated using formula (24) for $p = A$. This expression is a function of basic risk estimates. Formula (19) can be used to calculate $\hat{\sigma}_A$, and also to calculate $\hat{\bar{\sigma}}_A$ using the benchmark returns. Formula (20) can be used to calculate $\hat{c}_A$.

Information Ratio

A related statistic is the *information ratio*. This is the difference of the estimated mean portfolio return and the estimated mean benchmark return, divided by the estimated active risk:

$$\widehat{ir}_p := \frac{\hat{\mu}_p - \hat{\bar{\mu}}_p}{\hat{\sigma}_p^{Exc}} \quad . \quad (25)$$

This measure is a risk-adjusted return. It tells us how much the portfolio management has added on top of the benchmark, per unit of active risk. The annual information ratio $\widehat{ir}_A$ can be calculated using formula (25) for $p = A$. Use formula (18) for $\hat{\mu}_A$ and $\hat{\bar{\mu}}_A$, and (24) for $\hat{\sigma}_A^{Exc}$.

Regression Beta and Alpha

Two other common statistics, for which the annual variant can be easily estimated using estimates of the annual basic measures, are the *regression beta* and *alpha*:

$$\hat{\beta}_p := \frac{\hat{c}_p}{\hat{\bar{\sigma}}_p^2} \quad . \quad (26)$$

$$\hat{\alpha}_p := \hat{\mu}_p - \hat{\beta}_p \cdot \hat{\bar{\mu}}_p = \hat{\mu}_p - \frac{\hat{c}_p}{\hat{\bar{\sigma}}_p^2} \cdot \hat{\bar{\mu}}_p \quad . \quad (27)$$

The *beta* measures the tendency of the portfolio to move together with its benchmark. A beta around 1 indicates that the portfolio is highly correlated with the benchmark, and with a similar volatility. A beta near 0 indicates either that the portfolio and benchmark have a low correlation, or that the portfolio is less volatile than the benchmark.

The *alpha* measures the portfolio's additional return on top of the expected return given its exposure to the benchmark as represented by the beta.

Both statistics are functions of estimated statistics for which a formula for the annual variant has been derived earlier in this article.

More detailed information about the above statistics is available in the literature on performance measurement. See for example Bacon (2004).[9] This work also covers various other risk-related statistics. Many of these statistics can be written as functions of the basic risk measures. For all these statistics, annual variants can be estimated in a similar way as shown above for the active risk, information ratio, beta, and alpha. This way, a wide range of risk measures and related statistics is covered.

Table 6: Annual Estimated Risk Measures and Related Statistics in Various Scenarios

	Scenarios							
Annual Active risk	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Monthly	2.76%	8.27%	8.27%	8.27%	0.33%	0.89%	0.89%	0.89%
Annual, simple method	9.55%	28.65%	28.65%	28.65%	1.15%	3.10%	3.10%	3.10%
Annual, correct method	9.83%	29.75%	35.05%	23.87%	1.19%	3.26%	4.06%	2.78%
Annual Information ratio								
Monthly	0.04	0.01	0.06	-0.06	0.30	0.11	0.56	-0.56
Annual, simple method	0.13	0.04	0.21	-0.21	1.04	0.39	1.94	-1.94
Annual, correct method	0.13	0.04	0.21	-0.21	1.04	0.38	1.79	-1.78
Annual Beta								
Monthly	0.05	0.05	0.05	0.05	0.99	0.99	0.99	0.99
Annual, simple method	0.05	0.05	0.05	0.05	0.99	0.99	0.99	0.99
Annual, correct method	0.05	0.05	0.05	0.05	1.00	1.00	1.04	0.93
Annual Alpha								
Monthly	0.29%	0.29%	1.93%	-1.93%	0.10%	0.10%	0.52%	-0.52%
Annual, simple method	3.48%	3.48%	23.10%	-23.10%	1.23%	1.23%	6.20%	-6.20%
Annual, correct method	3.54%	3.54%	25.81%	-20.76%	1.24%	1.23%	6.41%	-6.02%

Numerical Examples

Table 6 considers the same eight scenarios (a)-(h) used earlier in Table 1 and Table 2. For each scenario this table provides a comparison of the estimated active risk, information ratio, regression beta and alpha, calculated using (24)-(27), for both the monthly and annual variant. In addition, the annual variant is based on the annual basic measures estimated using either the simple method (multiply by T or $\sqrt{T}$) or the correct method (see formulas (18)-(20)).

When looking at the resulting numbers for active risk, the differences between the simple and the correct method are largest when both the mean and standard deviation are large. This is similar to the results in Table 2 for the standard deviation.

For the information ratio, the results of the simple and correct method are mostly similar, unless high values occur for the mean, standard deviation and covariance. This is mainly the case in scenarios (g) and (h).

For the beta, when the simple method is used the annual beta is the same as the monthly beta. This is because the multiplication factor is T for both the numerator and denominator in (26). When the correct method is used, a difference occurs when the portfolio and benchmark series have a large covariance, but have a different mean. This difference in mean causes a difference in volatility, and, therefore, the annual beta changes a bit with respect to the monthly beta.

For the alpha, the correct method shows the largest differences compared to the simple method when the covariance between the portfolio and benchmark is small.

DOWNSIDE RISK MEASURES

Until this point, all annual risk measures could be estimated using two methods: a closed formula or the bootstrap method. Is this possible for any risk measure? Unfortunately, there exist measures for which it is not possible (or at least not straightforward) to derive a closed formula for the (estimated) annual variant. For those measures the bootstrap method has to be used instead. Below I will discuss two examples of such measures.

Downside Risk

One of the measures without closed formula for the annual version is the *downside risk*. Several variants of this measure exist. In this article, I use a very simple variant. The general definition of this downside risk variant σ_p^{Ds}, in any periodicity *p*, is:

$$\sigma_p^{Ds} := \sqrt{E\left[R_p^2 \cdot 1_{\{R_p<0\}}\right]} \quad . \quad (28)$$

Here, $1_{\{R_p<0\}}$ is a so-called *indicator-function*, which equals one if the periodic return is negative (R_p<0) and zero otherwise. In other words, σ_p^{Ds} is the standard deviation around zero, where the non-negative returns count as zero. Based on formula (28), an estimate of the downside risk, given a sample of realized periodic returns $r_{p,1}, \dots, r_{p,n}$, can be calculated as follows:

$$\hat{\sigma}_p^{Ds} := \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_{p,i}^2 \cdot 1_{\{r_{p,i}<0\}}} \quad . \quad (29)$$

The commonly used method to annualize a monthly downside risk is the same as for the standard deviation, *i.e.*, multiplying by $\sqrt{T}$. Note that this method is incorrect, *even* if you assume that monthly returns are summing up to the annual return! This is because the expression $R_A^2 \cdot 1_{\{R_A<0\}}$ cannot be expressed as a sum of similar monthly terms.

In addition, the presence of an indicator function makes it not straightforward to derive a closed formula for the annual downside risk, assuming that returns compound over time. Instead, the bootstrap method needs to be used to estimate the annual downside risk.

Value at Risk

Another risk measure, for which the bootstrap method proves to be powerful, is the *value at risk.* For a general periodicity *p*, let's consider the 5% value at risk $VaR_p^{5\%}$, which is defined as *minus* the 5% percentile of the probability distribution of R_p. This measure represents the potential loss (in percentage) of the portfolio in a future period, with a confidence level of 95% that the loss won't be even more. Under the assumption that past returns are representative for the probability distribution of the return in a future period, an estimate $\widehat{var}_p^{5\%}$ of the 5% value at risk can be found by calculating the 5% percentile of a dataset $r_{p,1}, \dots, r_{p,n}$ of realized periodic returns, with a minus sign. For instance, if the 5% percentile of the return data is -2%, then $\widehat{var}_p^{5\%}$ equals 2%, which represents a potential loss of 2 percent.

About annualizing the value at risk, similar remarks as for the downside risk can be made. The common method is to simply multiply by $\sqrt{T}$ and this is incorrect, even when monthly returns are summing up to the annual return.

Moreover, calculating the 5% value at risk of the annual return seems not straightforward without making assumptions on the distribution of the monthly returns. I therefore advise readers to use the bootstrap method to estimate the annual 5% value at risk.

Numerical Examples

For the downside risk and the 5% value at risk, for which closed formulas are not available, I will illustrate below how bootstrapping can be applied to estimate the annual variant using monthly returns.

Consider again the 4 scenarios (a)-(d) defined in Table 4, repeated in Table 7. For this analysis I have reused the

Table 7: Scenarios and Their Monthly Estimated Measures

	Scenarios			
Monthly measure	(a)	(b)	(c)	(d)
Mean	0.3%	0.3%	2%	-2%
Standard deviation	2%	6%	6%	6%
Downside risk	1.37%	4.39%	3.61%	5.66%
5% Value at risk	3.16%	10.07%	8.37%	12.37%

Table 8: Annual Estimated Measures, Using Various Techniques, in Various Scenarios

	Scenarios			
Annual Downside risk	(a)	(b)	(c)	(d)
Simple method ($\cdot\sqrt{T}$)	4.74%	15.22%	12.50%	19.60%
Bootstrap (∑)	3.24%	13.07%	5.22%	31.47%
Bootstrap (∏)	3.21%	12.36%	5.32%	26.96%
Annual 5% Value at risk				
Simple method ($\cdot\sqrt{T}$)	10.94%	34.89%	29.00%	42.86%
Bootstrap (∑)	8.11%	31.53%	11.13%	59.13%
Bootstrap (∏)	8.04%	29.33%	12.94%	47.02%

(∑) assuming summed returns
(∏) assuming compounded returns

custom set of monthly returns for each scenario, as described before in section "Bootstrapping." Next, I have estimated the monthly downside risk (using (29)) and the monthly 5% value at risk[10] based on the custom returns for each scenario. These estimates are given in Table 7, together with the monthly means and standard deviations.

Table 8 shows estimates of the annual downside risk and value at risk, after applying three different methods:

- the simple method (multiplying by $\sqrt{T}$);
- the bootstrap using summed returns in step 2;
- the bootstrap using compounded returns in step 2.

For both the downside risk and value at risk, the results using the simple method and the bootstrap with summed returns are very different. This is because, as noted before, the simple method by multiplying by $\sqrt{T}$ is *in*correct *even* when monthly returns sum up to the annual return. In this method the monthly estimate is multiplied by the same number regardless of the distribution of the monthly returns. In scenario (c), for instance, the monthly mean is quite high, 2%, with a standard deviation of 6 percent. Over a year, this gives a return distribution that is mostly positive, so that relatively low values occur for "downside" measures like the downside risk and the value at risk.

Multiplying by $\sqrt{T}$ highly overestimates the downside risk with 12.50% versus 5.22% as obtained by the bootstrap with summed returns. Also, for the value at risk a significant difference occurs: 29.00% versus 11.13 percent.

The opposite is true for scenario (d). Here, the monthly mean is quite negative, -2%, with a 6% standard deviation. Over a year, this gives a mostly negative return distribution so that the downside risk and value at risk become quite large. Multiplying by $\sqrt{T}$ underestimates these measures, compared to bootstrapping with summed returns: 19.60% versus 31.47% for the downside risk, and 42.86% versus 59.13% for the value at risk.

Scenarios (c) and (d) together show that multiplying by the same number $\sqrt{T}$, regardless of the return distribution, is in general incorrect. Such practice can either overestimate or underestimate the correct value of the annual downside measure.

The correct method to estimate the annual downside risk and value at risk is the bootstrap with compounded returns. The difference between this method and the bootstrap with summed returns is best illustrated by scenario (d). Here, the annual downside risk of 31.47%, obtained from the bootstrap with summed returns, significantly overestimates the value 26.96%, obtained from the bootstrap with compounded returns. For the value at risk the difference is also significant: 59.13% versus 47.02 percent.

One reason for these differences is that compounding makes the distribution of the annual return less negative than summing. A second reason is that the annual standard deviation is smaller when monthly returns are compounded rather than summed (see Table 5 for scenario (d)). These effects together decrease the potential for large losses when returns are compounded rather than summed, and, therefore, both the downside risk and the value at risk get significantly smaller. The bootstrap with compounded returns correctly reflects the effects of return compounding.

Since closed formulas to estimate the annual downside risk and value at risk based on monthly returns are not available, the bootstrap proves to be a useful method to estimate these measures. As a corollary, for any risk statistic which is a function of a downside risk measure and other measures discussed in this article, the annual variant can be estimated in a similar way as shown in the previous section for the active risk, information ratio, beta and alpha. For example, the *Sortino ratio* is a ratio using the downside risk in its denominator and a mean return in its numerator. Since the annual downside risk can be estimated using a bootstrap, and the mean return can be estimated using either a closed formula or a bootstrap, it follows that the annual Sortino ratio can be estimated by dividing the annual mean over the annual downside risk.

In the end, the methods in this article provide ways to estimate annual variants of a broad set of risk measures and related statistics.

CONCLUSION

In this article I have researched the methods to estimate annual risk measures based on realized monthly returns, thereby extending the work of Kaplan. The most commonly used method in the industry is simply multiplying by a constant number: either the number of months in a year T or its square root $\sqrt{T}$. Assuming that returns are compounded over time, rather than summed, I have shown that this approach is mathematically incorrect. For some risk measures, like the downside risk and the value at risk, this method is *even* incorrect under the assumption of summed returns.

For the mean, standard deviation and covariance (the so-called "basic measures"), I have presented two methods to correctly estimate the annual variants: one method which results in closed formulas, and one which involves simulation: bootstrapping. Several risk-related statistics are functions of the basic measures. Calculating an annual estimate of such a derived statistic is straightforward given that estimates of the underlying annual measures are available. I have shown how this can be done for the active risk, information ratio, regression beta, and alpha.

For measures for which an explicit formula to calculate an estimation is not available, bootstrapping is the method to use. I have illustrated the application of bootstrapping for two of such measures: the downside risk and the value at risk. If a risk-related statistic can be derived from measures for which an annual estimation is available (either via a closed formula or via bootstrapping), an annual estimation is available for this statistic as well. This way, the methods in this article cover a wide palette of risk measures and related statistics.

The numerical examples in this article show that the resulting estimations can differ significantly between the simple (incorrect) method and the correct method, whether a closed formula or bootstrapping is used. One of the most prominent observations is that for many measures, the differences between methods are large when the means of the given series of monthly returns are large (positively or negatively).

ENDNOTES

[1] Kaplan, Paul D., "What's Wrong with Multiplying by the Square Root of Twelve," *The Journal of Performance Measurement*, Winter 2012/2013.

[2] The estimates (5) and (6) take Bessel's correction into account. See: https://en.wikipedia.org/wiki/Bessel's_correction.

[3] Throughout this article I work with monthly return, so $T \coloneqq 12$, although the presented methods are valid for any periodicity assuming a fixed number of periods in a year. For instance, if you wish to apply these methods for weekly returns, use $T \coloneqq 52$.

[4] Note, however, that in the field of investment performance it is uncommon to annualize monthly returns. Because the method assumes a year in which all T months have the same return as the given month, the resulting annual return might be misleading. Nevertheless, if you want to annualize a monthly return, formula (21) gives the correct method.

[5] The term "annualization" suggests a direct transformation from a monthly (unannualized) number to an annual (annualized) number. Because there exist measures, like the standard deviation, for which the correct method to estimate the annual variant is not a direct transformation, but needs other input in addition to T and the monthly number, I prefer to use the term "annual measure" rather than "annualized measure."

[6] If your dataset consists of pairs of monthly portfolio and benchmark returns, and both are needed for the target measure, then the bootstrap algorithm randomly selects *pairs* of monthly returns in step 1 (instead of single returns), and calculates both annual portfolio returns and annual benchmark returns in step 2.

[7] For integrity reasons, this portfolio is not further specified. Furthermore, to prevent the portfolio being identifiable from the data, the monthly returns are ordered by size instead of chronologically. Lastly, a small random noise term has been added to the returns.

[8] Note, however, that if the excess returns were *geometric* excess returns, then the monthly excess returns do compound to an annual excess return, and formula (19) *does* apply.

[9] Bacon, Carl R., Practical Portfolio Performance Measurement and Attribution, Wiley, London, 2004, pp. 53-80.

[10] To calculate the 5% percentile of a set of returns I use the Excel function PERCENTILE.EXC.

APPENDIX A – PROOF OF THE SIMPLE METHOD TO ANNUALIZE THE BASIC MEASURES

In this appendix, I give the mathematical proofs of formulas (8)-(10) to annualize the three basic measures using the simple method.

Consider all variables as defined in the main text, and assume that the annual return is made up by *summing* the monthly returns:

$$R_A := \sum_{t=1}^{T} R_{m,t} \quad . \qquad (30)$$

and similarly for the benchmark.

<u>Mean</u>

Equation (8) for the annualized mean can be proven by using the law from probability that the expectation of a sum of random variables equals the sum of the expectations of the random variables:

$$\mu_A = E[R_A] = E\left[\sum_{t=1}^{T} R_{m,t}\right] = \sum_{t=1}^{T} E[R_{m,t}] = T \cdot E[R_m] = T \cdot \mu_m \quad . \qquad (31)$$

<u>Standard Deviation</u>

To annualize the annual standard deviation, I will first show how to annualize the *variance*. This is by definition the square of the standard deviation. Here I use the law that the variance of a sum of *independent* random variables equals the sum of the individual variances:

$$\sigma_A^2 = Var[R_A] = Var\left[\sum_{t=1}^{T} R_{m,t}\right] = \sum_{t=1}^{T} Var[R_{m,t}] = T \cdot Var[R_m] = T \cdot \sigma_m^2 \quad . \qquad (32)$$

Equation (9) for the standard deviation follows by taking the square root of the result of (32):

$$\sigma_A = Std[R_A] = \sqrt{Var[R_A]} = \sqrt{T} \cdot \sigma_m . \qquad (33)$$

<u>Covariance</u>

The covariance of R_A and $\bar{R}_A$ can be written out as follows, using first a calculation property of the covariance, and afterwards the summation law for the expectation again:

$$c_A = Cov[R_A, \bar{R}_A] = Cov\left[\sum_{t=1}^{T} R_{m,t}, \sum_{t=1}^{T} \bar{R}_{m,t}\right]$$

$$= E\left[\sum_{t=1}^{T} R_{m,t} \cdot \sum_{t=1}^{T} \bar{R}_{m,t}\right] - E\left[\sum_{t=1}^{T} R_{m,t}\right] \cdot E\left[\sum_{t=1}^{T} \bar{R}_{m,t}\right]$$

$$= E\left[\sum_{t=1}^{T}\sum_{s=1}^{T} R_{m,t} \cdot \bar{R}_{m,s}\right] - \sum_{t=1}^{T} E[R_{m,t}] \cdot \sum_{t=1}^{T} E[\bar{R}_{m,t}]$$

$$= \sum_{t=1}^{T}\sum_{s=1}^{T}\left(E[R_{m,t}, \bar{R}_{m,s}] - E[R_{m,t}] \cdot E[\bar{R}_{m,s}]\right) = \sum_{t=1}^{T}\sum_{s=1}^{T} Cov[R_{m,t}, \bar{R}_{m,s}] \quad . \qquad (34)$$

Because the covariance of two independent random variables is zero, it follows that

$$Cov\left[R_{m,t}, \bar{R}_{m,s}\right] = 0$$

for $t \neq s$, and the above sum reduces to:

$$c_A = \sum_{t=1}^{T} Cov\left[R_{m,t}, \bar{R}_{m,t}\right] = T \cdot Cov[R_m, \bar{R}_m] = T \cdot c_m \quad . \quad (35)$$

This proves (10).

APPENDIX B – PROOF OF THE CORRECT METHOD FOR THE ANNUAL BASIC MEASURES

In this appendix, I give the mathematical proofs of the correct formulas (15)–(17) for the annual basic measures. Assume that the annual return is made up by *compounding* the monthly returns:

$$R_A = \prod_{t=1}^{T}\left(1 + R_{m,t}\right) - 1 \quad , \quad (36)$$

and similarly for the benchmark.

<u>Mean</u>

To calculate the correct formula for the annual mean, I use the law from probability that the expectation of a product of *independent* random variables equals the product of the expectations of the random variables:

$$\mu_A = E[R_A] = E\left[\prod_{t=1}^{T}\left(1 + R_{m,t}\right) - 1\right] =$$

$$E\left[\prod_{t=1}^{T}\left(1 + R_{m,t}\right)\right] - 1$$

$$= \prod_{t=1}^{T} E\left[1 + R_{m,t}\right] - 1 = (E[1 + R_m])^T -$$

$$1 = (1 + \mu_m)^T - 1 \quad . \quad (37)$$

This proves equation (15).

<u>Standard Deviation</u>

Kaplan's article features a proof of equation (16) for the annual standard deviation. For completeness, I will provide a similar derivation.

To calculate the annual standard deviation, I will first derive a formula for the annual variance. According to a calculation law of probability, the monthly variance σ_m^2 can be written as:

$$\sigma_m^2 = Var[R_m] = E[R_m^2] - \mu_m^2 \quad . \quad (38)$$

so

$$E[R_m^2] = \sigma_m^2 + \mu_m^2. \quad . \quad (39)$$

Since adding a constant to a random variable does not change its variance, the annual variance σ_A^2 can be written as:

$$\sigma_A^2 = Var[R_A] = Var[1 + R_A] =$$

$$E[(1 + R_A)^2] - E[1 + R_A]^2$$

$$= E\left[\prod_{t=1}^{T}\left(1 + R_{m,t}\right)^2\right] - \left(E\left[\prod_{t=1}^{T}\left(1 + R_{m,t}\right)\right]\right)^2 . \quad (40)$$

Again, since the expectation of a product of independent random variables equals the product of the expectations, we find:

$$\sigma_A^2 = \prod_{t=1}^{T} E\left[1 + R_{m,t}\right]^2 - \left(\prod_{t=1}^{T} E\left[1 + R_{m,t}\right]\right)^2 =$$

$$(E[(1 + R_m)^2])^T - (1 + \mu_m)^{2T}$$

$$= (1 + 2\mu_m + E[R_m^2])^T - (1 + \mu_m)^{2T} \quad . \quad (41)$$

Substituting (39) in (41) gives:

$$\sigma_A^2 = (1 + 2\mu_m + \sigma_m^2 + \mu_m^2)^T - (1 + \mu_m)^{2T} =$$

$$(\sigma_m^2 + (1 + \mu_m)^2)^T - (1 + \mu_m)^{2T} \quad . \quad (42)$$

Because the annual standard deviation is the square root of the annual variance:

$$\sigma_A = \sqrt{(\sigma_m^2 + (1 + \mu_m)^2)^T - (1 + \mu_m)^{2T}} . \quad (43)$$

Which proves equation (16).

<u>Covariance</u>

The derivation of the annual covariance is very similar to the variance. The monthly covariance equals:

$$c_m = Cov[R_m, \bar{R}_m] = E[R_m \cdot \bar{R}_m] - \mu_m \bar{\mu}_m \,. \quad (44)$$

so

$$E[R_m \cdot \bar{R}_m] = c_m + \mu_m \bar{\mu}_m. \quad (45)$$

The annual covariance equals:

$$c_A = Cov[R_A, \bar{R}_A] = Cov[1 + R_A, 1 + \bar{R}_A]$$
$$= E[(1 + R_A)(1 + \bar{R}_A)] - E[1 + R_A] \cdot E[1 + \bar{R}_A]$$
$$= E\left[\prod_{t=1}^{T}(1 + R_{m,t})(1 + \bar{R}_{m,t})\right] - E\left[\prod_{t=1}^{T}(1 + R_{m,t})\right]$$
$$\cdot E\left[\prod_{t=1}^{T}(1 + \bar{R}_{m,t})\right]. \quad (46)$$

Using again the product rule of expectations of independent random variables gives:

$$c_A = \prod_{t=1}^{T} E[(1 + R_{m,t})(1 + \bar{R}_{m,t})] -$$
$$\prod_{t=1}^{T} E[1 + R_{m,t}] \cdot \prod_{t=1}^{T} E[1 + \bar{R}_{m,t}]$$
$$= (E[(1 + R_m)(1 + \bar{R}_m)])^T - (1 + \mu_m)^T (1 + \bar{\mu}_m)^T$$
$$= (1 + \mu_m + \bar{\mu}_m + E[R_m \cdot \bar{R}_m])^T - (1 + \mu_m)^T (1 + \bar{\mu}_m)^T \,. \quad (47)$$

Substituting (45) in (47) gives:

$$c_A = (1 + \mu_m + \bar{\mu}_m + c_m + \mu_m \bar{\mu}_m)^T -$$
$$(1 + \mu_m)^T (1 + \bar{\mu}_m)^T$$
$$= \left(c_m + (1 + \mu_m)(1 + \bar{\mu}_m)\right)^T - (1 + \mu_m)^T (1 + \bar{\mu}_m)^T \,. \quad (48)$$

This concludes the proof of equation (17).

ACKNOWLEDGEMENT

I thank my colleagues Bas Leerink and Herbert Beltman for their extensive review and feedback, which has enhanced the article considerably.